JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **149**, 255–270 (1990)

Optimality of Differentiable, Vector-Valued n-Set Functions*

LAI-JIU LIN

Department of Mathematics, National Changhua University of Education, Changhua, Taiwan, Republic of China

Submitted by E. Stanley Lee

Received August 17, 1988

The necessary and sufficient conditions for the existence of an optimal solution of a vector-valued, n-set functions optimization problem is obtained in this paper.

1. INTRODUCTION

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable and $F: \mathscr{S} \to \mathbb{R}^m$, $G: \mathscr{S} \to \mathbb{R}^p$ defined on a convex subfamily $\mathscr{S}$ of $\Gamma^n = \Gamma \times \cdots \times \Gamma$, we consider an optimization problem as

$$\text{minimize } F(\Omega_1, ..., \Omega_n) \text{ subject to } (\Omega_1, ..., \Omega_n) \in \mathscr{S}$$
$$\text{and } G(\Omega_1, ..., \Omega_n) \leqq 0. \tag{P}$$

In [12], Morris first considered the general theory of real-valued set functions of a single set. He showed the necessary and sufficient conditions for a constrained local minimum of real-valued set functions of a single set. Following the Morris setting, Chou *et al.* [1] characterized the proper efficient solutions for the problem (P) in terms of a optimal solution for associated scalar problems. In [13], Tanaka considered the Pareto optimization of (P) and showed the necessary and sufficient conditions for the existence of the local Pareto minimum to (P). In [1, 6, 7, 12, 13], the optimization problem has remained confined to set functions of a single set. In [4], Corley first developed the general theory for n-set functions and gave the concepts of partial derivative and derivative of n-set function. In this paper, we prove the Farkas–Minkowski type theorem for vector-valued n-set functions. Using this result we establish the necessary and suf-

* This research was supported by the National Science Council of the Republic of China.

0022-247X/90 $3.00

ficient conditions for the existence of weak local minimum to (P) in terms of the derivatives of vector-valued n-set functions involved. Because the Pareto minimum to (P) is also the weak minimum, but the converse is not true, hence our results and methods are quite different from Theorems 1 and 2 of [13]. When the objective functions are real-valued, our results reduce to Theorems 3.7, 3.8, and 4.7 of [4].

2. Preliminary

Throughout this paper, we assume (X, Γ, μ) is a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable and let $\Gamma^n = \Gamma \times \cdots \times \Gamma = \{(\Omega_1, ..., \Omega_n) \mid \Omega_i \in \Gamma,\ i = 1, ..., n\}$. We define a pseudometric d on Γ^n as

$$d[(\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)] = \left\{ \sum_{i=1}^{n} [\mu(\Omega_i \Delta \Lambda_i)]^2 \right\}^{1/2},$$

$\Omega_i, \Lambda_i \in \Gamma$, $i = 1, ..., n$, where $\Omega_i \Delta \Lambda_i$ denotes symmetric difference for Ω_i and Λ_i. Essentially $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n)$ will be regarded as equivalent if $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) = 0$. We see that Γ^n is only a semialgebra but not a σ-algebra. For $f \in L_1(X, \Gamma, \mu)$ and $\Omega \in \Gamma$, the integral $\int_\Omega f\, d\mu$ will be denoted by $\langle f, \chi_\Omega \rangle$, where χ_Ω denotes the characteristic function of Ω. We introduce the following notations for the vectors in the m-dimensional Euclidean space $\mathbb{R}^m$. For two vectors $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ in $\mathbb{R}^m$,

(i) $x < y$ iff $x_i < y_i$ for all $i = 1, ..., m$.

(ii) $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, ..., m$ and $x \neq y$.

(iii) $x \leqq y$ iff $x_i \leq y_i$ for all $i = i, ..., m$.

The zero vector $(0, ..., 0)$ in $\mathbb{R}^m$ is denoted by 0 and the nonnegative orthant is denoted by $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x \geqq 0\}$. We denote by $B(\mathbb{R}^m, \mathbb{R}^p)$, the set of all continuous linear operators from $\mathbb{R}^m$ to $\mathbb{R}^p$ and

$$B^+(\mathbb{R}^m, \mathbb{R}^p) = \{w \in B(\mathbb{R}^m, \mathbb{R}^p) \mid w(\mathbb{R}^m_+) \subset \mathbb{R}^p_+\}.$$

Definition 2.1. Let $A \subset \mathbb{R}^m$, a point $y_0 \in A$ is said to be a weak minimum of A, denoted by $y_0 \in$ w-min A if there does not exist y in A such that $y < y_0$, and $y_0 \in A$ is said to be a minimum of A if $y_0 \leqq y$ for all $y \in A$.

Definition 2.2. A set function $F\colon \Gamma \to \mathbb{R}$ is differentiable at $\Omega \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$, the derivative of F at Ω such that

$$F(\Lambda) = F(\Omega) + \langle f, \chi_\Lambda - \chi_\Omega \rangle + d(\Omega, \Lambda)\, E(\Omega, \Lambda),$$

where

$$\lim_{d(\Omega, \Lambda) \to 0} E(\Omega, \Lambda) = 0.$$

DEFINITION 2.3. Let $F: \Gamma^n \to \mathbb{R}$ and $(\Omega_1, ..., \Omega_n) \in \Gamma^n$. Then F is said to have a partial derivative at $(\Omega_1, ..., \Omega_n)$ with respect to Λ_i if the set function

$$H(\Lambda_i) = F(\Omega_1, ..., \Omega_{i-1}, \Lambda_i, \Omega_{i+1}, ..., \Omega_n)$$

has derivative h_{Ω_i} at Ω_i. In this case we define the ith partial derivative of F at $(\Omega_1, ..., \Omega_n)$ to be $f^i_{\Omega_1, ..., \Omega_n} = h_{\Omega_i}$.

Now, we define the derivative of vector-valued n-set functions.

DEFINITION 2.4. Let $\mathscr{S} \subset \Gamma^n$, $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ and $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$. Then F is said to be differentiable at $(\Omega_1, ..., \Omega_n)$ if the partials $f^{ij}_{\Omega_1, ..., \Omega_n}$, $i = 1, 2, ..., n$, of F_j exist for each $j = 1, 2, ..., m$ and satisfy

$$\begin{aligned} F(\Lambda_1, ..., \Lambda_n) &= F(\Omega_1, ..., \Omega_n) \\ &\quad + \left(\sum_{i=1}^{n} \langle f^{i1}_{\Omega_1, ..., \Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{im}_{\Omega_1, ..., \Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\ &\quad + W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)), \quad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}, \end{aligned}$$

where

$$\frac{W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))}{d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))} \to 0$$

as $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) \to 0$.

Throughout the paper if $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ and $G = (G_1, ..., G_p): \mathscr{S} \to \mathbb{R}^p$ are differentiable at $(\Omega_1, ..., \Omega_n)$, we will denote $f^{ij}_*, ..., g^{ij}_*$ the ith partial derivatives of F_j and G_j at $(\Omega_1, ..., \Omega_n)$, respectively.

Similar to [12, Proposition 3.2 and Lemma 3.3], for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exists sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Lambda \setminus \Omega} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1 - \lambda) \chi_{\Omega \setminus \Lambda} \tag{1}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Lambda + (1 - \lambda) \chi_\Omega, \tag{2}$$

where w^* stands for the w^*-convergence. The sequence $\{V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ satisfying (1) and (2) is called the Morris sequence associated with $(\Omega, \Lambda, \lambda)$.

DEFINITION 2.5. A subfamily $\mathscr{S}$ of Γ^n is convex if given $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$, where N is the set of natural numbers.

DEFINITION 2.6. A set function $F: \mathscr{S} \to \mathbb{R}^m$ is called $\mathbb{R}_+^m$-convex on a convex subfamily $\mathscr{S}$ of Γ^n if for each $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}}\, F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1 - \lambda)\, F(\Omega_1, ..., \Omega_n).$$

EXAMPLE. If $F: \Gamma^n \to \mathbb{R}^m$ is convex on Γ^n, then the subfamily

$$\mathscr{S} = \{(\Omega_1, ..., \Omega_n) \in \Gamma^n \mid F(\Omega_1, ..., \Omega_n) < 0\}$$

is a convex subfamily of Γ^n.

3. MAIN RESULTS

DEFINITION 3.1. Let $\mathscr{S}$ be a nonempty subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$. Then $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$ if $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$ if there exists $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ satisfying $d[(\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)] < \delta$.

THEOREM 1. *Let $\mathscr{S}$ be a convex subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$ be a $\mathbb{R}_+^m$-convex set function. If $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$, then $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$.*

Proof. Since $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$, there exists $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ with $d[(\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)] < \delta$. Fix $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$. Then by the convexity of F on the convex subfamily $\mathscr{S}$ of Γ^n, for any $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}}\, F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1 - \lambda)\, F(\Omega_1, ..., \Omega_n).$$

Since

$$\begin{aligned}\lim_{k\to\infty} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) &= \lim_{k\to\infty} \left\{\sum_{k=1}^{n} [\mu(V_i^k(\lambda)\,\Delta\Omega_i)]^2\right\}\\ &= \lim_{k\to\infty} \left\{\sum_{i=1}^{n} \|\chi_{V_i^k(\lambda)} - \chi_{\Omega_i}\|_{L_1}^2\right\}^{1/2}\\ &= \left\{\sum_{i=1}^{n} \lambda^2 \|\chi_{A_i} - \chi_{\Omega_i}\|_{L_1}^2\right\}^{1/2}\\ &= \lambda\left\{\sum_{i=1}^{n} [\mu(A_i\,\Delta\Omega_i)]^2\right\}^{1/2}\\ &= \lambda d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)),\end{aligned}$$

there exists $r > 0$ and a natural number M such that

$$d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < \delta \quad \text{for} \quad 0 < \lambda < r \quad \text{and} \quad k \geq M.$$

Hence

$$F(\Omega_1, ..., \Omega_n) \leqq F(V_1^k(\lambda), ..., V_n^k(\lambda)) \quad \text{for} \quad 0 < \lambda < r \quad \text{and} \quad k \geq M.$$

From this, we obtain

$$\begin{aligned}F(\Omega_1, ..., \Omega_n) &\leqq \overline{\lim_{k\to\infty}}\ F(V_1^k(\lambda), ..., V_n^k(\lambda))\\ &\leqq \lambda F(A_1, ..., A_n) + (1-\lambda)\, F(\Omega_1, ..., \Omega_n)\end{aligned}$$

for all $0 < \lambda < r$. This implies

$$F(\Omega_1, ..., \Omega_n) \leqq F(A_1, ..., A_n).$$

Since $(A_1, ..., A_n) \in \mathscr{S}$ is arbitrary, this shows that $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$. Q.E.D.

In order to obtain the main result, we need the following Farkas–Minkowski type theorem for n-set functions.

THEOREM 2. *Let $\mathscr{S}$ be a convex subfamily of Γ^n,*

$$F = (F_1, ..., F_m) : \mathscr{S} \to \mathbb{R}^m \text{ be } \mathbb{R}_+^m\text{-convex}$$

and

$$G = (G_1, ..., G_p) : \mathscr{S} \to \mathbb{R}^p \text{ be } \mathbb{R}_+^p\text{-convex}.$$

If the system

$$\begin{cases} F(\Omega_1, ..., \Omega_n) < 0 \\ G(\Omega_1, ..., \Omega_n) < 0 \end{cases}$$

has no solution in $\mathscr{S}$, *then there exists* $u = (u_1, ..., u_m) \in \mathbb{R}^m_+$, $v = (v_1, ..., v_p) \in \mathbb{R}^p_+$, $(u, v) \neq (0, 0)$ *such that*

$$\sum_{i=1}^{m} u_i F_i(\Lambda_1, ..., \Lambda_n) + \sum_{i=1}^{p} v_i G_i(\Lambda_1, ..., \Lambda_n) \geqq 0$$

for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$.

Proof. Let $A = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^p \mid \text{there exists } (\Omega_1, ..., \Omega_n) \in \mathscr{S} \text{ such that } F(\Omega_1, ..., \Omega_n) < y \text{ and } G(\Omega_1, ..., \Omega_n) < z\}$. It is obvious that A does not contain the origin of $\mathbb{R}^m \times \mathbb{R}^p$. To show that A is convex in $\mathbb{R}^m \times \mathbb{R}^p$, let (y, z) and $(\bar{y}, \bar{z})$ be in A, then there exist $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ such that

$$F(\Omega_1, ..., \Omega_n) < y, \qquad G(\Omega_1, ..., \Omega_n) < z$$

and

$$F(\Lambda_1, ..., \Lambda_n) < \bar{y}, \qquad G(\Lambda_1, ..., \Lambda_n) < \bar{z}.$$

It follows from the convexity of F and G on the convex subfamily $\mathscr{S}$ of Γ^n, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$, and

$$\begin{aligned} \overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) &\leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1-\lambda) F(\Omega_1, ..., \Omega_n) \\ &< \lambda \bar{y} + (1-\lambda) y \end{aligned}$$

and

$$\begin{aligned} \overline{\lim_{k \to \infty}} G(V_1^k(\lambda), ..., V_n^k(\lambda)) &\leqq \lambda G(\Lambda_1, ..., \Lambda_n) + (1-\lambda) G(\Omega_1, ..., \Omega_n) \\ &< \lambda \bar{z} + (1-\lambda) z. \end{aligned}$$

Therefore, there exists an integer $M > 0$ such that

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < \lambda \bar{y} + (1-\lambda) y$$

and

$$G(V_1^k(\lambda), ..., V_n^k(\lambda)) < \lambda \bar{z} + (1-\lambda) z$$

for $k \geqslant M$. Hence

$$\lambda(\bar{y}, \bar{z}) + (1-\lambda)(y, z) = (\lambda\bar{y} + (1-\lambda) y, \lambda\bar{z} + (1-\lambda) z) \in A.$$

It is obvious that A has a nonempty interior. Since $(0, 0) \notin A$, it follows from the separation theorem that there exist $u = (u_1, ..., u_m) \in \mathbb{R}^m$, $v = (v_1, ..., v_p) \in \mathbb{R}^m$ such that $(u, v) \neq (0, 0)$ and

$$\sum_{i=1}^{m} u_i y_i + \sum_{i=1}^{p} v_i z_i \geq 0 \qquad \text{for all} \quad (y, z) \in A, \tag{3}$$

where $Y = (y_1, ..., y_m)$, $z = (z_1, ..., z_p)$.

Following a similar argument as in Lemma 3.1 [1] we can show that $u \geqq 0$, $v \geqq 0$, and

$$\sum_{i=1}^{m} u_i F_i(A_1, ..., A_n) + \sum_{i=1}^{p} v_i G_i(A_1, ..., A_n) \geqq 0$$

for all $(A_1, ..., A_n) \in \mathscr{S}$. Q.E.D.

DEFINITION 3.2. Let $\mathscr{S}$ be a nonempty subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$. Then $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$ is called a weak local minimum of F on $\mathscr{S}$ if there exists $\delta > 0$ such that there does not exist $(A_1, ..., A_n) \in \mathscr{S}$ with $d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) < \delta$ and $F(A_1, ..., A_n) < F(\Omega_1, ..., \Omega_n)$. $(\Omega_1, ..., \Omega_n)$ is called a weak minimum of F on $\mathscr{S}$ if there does not exist $(A_1, ..., A_n) \in \mathscr{S}$ such that $F(A_1, ..., A_n) < F(\Omega_1, ..., \Omega_n)$.

Remark. It follows from Definitions 3.1 and 3.2 that if $F: \mathscr{S} \to \mathbb{R}$ and $(\Omega_1, ..., \Omega_n)$ is a weak local minimum of F on $\mathscr{S}$, then it is a local minimum of F on $\mathscr{S}$.

Applying Theorem 2, we have the following theorem.

THEOREM 3. *Let $\mathscr{S}$ be a convex subfamily of Γ^n and $F = (F_1, ..., F_m)$: $\mathscr{S} \to \mathbb{R}^m$, $G = (G_1, ..., G_p)$; $\mathscr{S} \to \mathbb{R}^p$ are differentiable at $(\Omega_1, ..., \Omega_n)$. Assume that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to problem* (P). *Then there exists nonzero element*

$$(\lambda, u) = ((\lambda_1, ..., \lambda_m), (u_1, ..., u_p)) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+$$

such that

$$\sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n) = 0$$

and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j < f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} > + \sum_{j=1}^{m} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geq 0$$

for all $(A_1, ..., A_n) \in \mathscr{S}$.

Proof. Define

$$H_1(A_1, ..., A_n) = \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right)$$

$$H_2(A_1, ..., A_n) = \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) + G(\Omega_1, ..., \Omega_n).$$

It is obvious that H_1 is $\mathbb{R}_+^m$-convex and H_2 is $\mathbb{R}_+^p$-convex.

We claim that the system

$$\begin{cases} H_1(A_1, ..., A_n) < 0 \\ H_2(A_1, ..., A_n) < 0 \end{cases} \tag{4}$$

has no solution. If $(A_1, ..., A_n) \in \mathscr{S}$ were a solution of (4), fix $\lambda \in [0, 1]$; since $\mathscr{S}$ is a convex subfamily of Γ^n, it follows that there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with (Ω_i, A_i, λ) for $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$. Then by the differentiability of F and G at $(\Omega_1, ..., \Omega_n)$, we would have

$$\begin{aligned} F(V_1^k(\lambda), ..., V_n^k(\lambda)) = F(\Omega_1, ..., \Omega_n) & \\ + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle \right) & \\ + E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) & \qquad (5) \end{aligned}$$

and

$$\begin{aligned} G(V_1^k(\lambda), ..., V_n^k(\lambda)) = G(\Omega_1, ..., \Omega_n) & \\ + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle \right) & \\ + \tilde{E}((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)), & \qquad (6) \end{aligned}$$

where $E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$ and $\tilde{E}((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$ are $o(d[(V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)])$. If we express

$$E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) = (E_1((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)), ..., E_m((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))).$$

Then $E_i((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$ is $o(d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)))$ for each $i = 1, ..., m$. Therefore for each $\varepsilon > 0$ and $i = 1, ..., m$, there exists $r > 0$ such that $|E_i((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))| \leq \varepsilon d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$ for $d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < r$. Let $\delta = r/d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))$. Then $\lim_{k \to \infty} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) = \lambda d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))$ implies that for $\lambda < \delta$ and for sufficiently large k, we have $d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < r$. Hence

$$|E_i((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))| \leq \varepsilon d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$$

for each $i = 1, ..., m$. This shows that $\overline{\lim}_{k \to \infty} E_i((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$ is $o(\lambda)$ for each $i = 1, ..., m$ and therefore

$$\begin{aligned}
&\overline{\lim_{k \to \infty}}\ E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \\
&\quad = (\overline{\lim_{k \to \infty}}\ E_1((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)), ..., \\
&\quad\quad \overline{\lim_{k \to \infty}}\ E_m((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))) \\
&\quad = o(\lambda)
\end{aligned} \tag{7}$$

Similarly $\overline{\lim}_{k \to \infty} \tilde{E}((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) = o(\lambda)$. It follows from (5), (6), and (7) that

$$\begin{aligned}
&\overline{\lim_{k \to \infty}}\ F(V_1^k(\lambda), ..., V_n^k(\lambda)) \\
&\quad = F(\Omega_1, ..., \Omega_n) + \lambda \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + o(\lambda) \\
&\quad = F(\Omega_1, ..., \Omega_n) + \lambda H_1(\Lambda_1, ..., \Lambda_n) + o(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\lim_{k \to \infty}}\ G(V_1^k(\lambda), ..., V_n^k(\lambda)) \\
&\quad = G(\Omega_1, ..., \Omega_n) + \lambda \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + o(\lambda) \\
&\quad = (1 - \lambda)\, G(\Omega_1, ..., \Omega_n) + \lambda H_2(\Lambda_1, ..., \Lambda_n) + o(\lambda).
\end{aligned}$$

Since $H_1(\Lambda_1, ..., \Lambda_n) < 0$ and $H_2(\Lambda_1, ..., \Lambda_n) < 0$, for any $\delta > 0$, we can choose a small $\lambda' > 0$ and a natural number k such that

$$F(V_1^k(\lambda'), ..., F_n^k(\lambda')) < F(\Omega_1, ..., \Omega_n)$$
$$G(V_1^k(\lambda'), ..., V_n^k(\lambda')) < (1-\lambda')\, G(\Omega_1, ..., \Omega_n) \leqq 0$$

and

$$d((V_1^k(\lambda'), ..., V_n^k(\lambda')), (\Omega_1, ..., \Omega_n)) < \delta.$$

This contradicts the assumption that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to (P). Hence system (4) does not have a solution. It follows from Theorem 2 that there exists a nonzero element $(\lambda, u) = ((\lambda_1, ..., \lambda_m), (u_1, ..., u_p)) \in \mathbb{R}_+^m \times \mathbb{R}_+^p$ such that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$+ \sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n) \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}. \tag{8}$$

Letting $(\Lambda_1, ..., \Lambda_n) = (\Omega_1, ..., \Omega_n)$ in (8), we obtain

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \geq 0.$$

Since $u \geqq 0$ and $G(\Omega_1, ..., \Omega_n) \leqq 0$, it must be

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \leq 0.$$

It then reduces to

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) = 0.$$

Then by (8), we get

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$+ \sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \geq 0$$

for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$. Q.E.D.

Remark. Since weak minimum is different from Pareto minimum, our result is different from Theorem 1 [13]. For $m=1$, Theorem 3 reduces to Theorem 3.7 [4].

If we give an additional condition of regularity for the inequality constraint, then we get

THEOREM 4. *In Theorem* 3, *if we assume further that there exists a* $(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) \in \mathscr{S}$ *such that*

$$G(\Omega_1, \ldots, \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i} - \Omega_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle \right) < 0,$$

then there exists $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ *such that*

$$w[G(\Omega_1, \ldots, \Omega_n)] = 0$$

and

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right)$$
$$+ w\left[\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right] < 0$$

fail to hold for any $(A_1, \ldots, A_n) \in \mathscr{S}$.

Proof. It follows from Theorem 3, that there exists nonzero $(\lambda, u) = ((\lambda_1, \ldots, \lambda_m), (u_1, \ldots, u_p)) \in \mathbb{R}_+^m \times \mathbb{R}_+^p$ such that

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, \ldots, \Omega_n) = 0$$

and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{m} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geqq 0, \tag{9}$$

for all $(A_1, \ldots, A_n) \in \mathscr{S}$.

By assumption, there exists $(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) \in \mathscr{S}$ such that

$$G(\Omega_1, \ldots, \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle \right) < 0.$$

If $\lambda = 0$, then $u \neq 0$ and $u \geqq 0$ and so $\sum_{i=1}^{p} u_i z_i > 0$ for all $z = (z_1, ..., z_p) \in \mathbb{R}^p$ and $z > 0$. Thus, by assumption, $\lambda = 0$, we should get

$$0 > \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j (g_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i}) \geq 0.$$

This is a contradiction; therefore $\lambda \neq 0$. Since $\lambda \geqq 0$ and $\lambda \neq 0$, we can choose $v = (v_1, ..., v_m) \in \mathbb{R}^m$ and $v > 0$ such that

$$\sum_{i=1}^{m} \lambda_i v_i = 1.$$

Define $w = (w_1, ..., w_m) \colon \mathbb{R}^p \to \mathbb{R}^m$ by

$$w(z) = \left(\sum_{i=1}^{p} u_i z_i \right) v,$$

where $z = (z_1, ..., z_p) \in \mathbb{R}^p$. Then $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ and $w[G(\Omega_1, ..., \Omega_n)] = [\sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n)] v = 0$. By (9), we obtain

$$\sum_{j=1}^{m} \lambda_j \Bigg[\sum_{i=1}^{n} \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle$$
$$+ w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \Bigg]$$
$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geq 0.$$

Since $\lambda \geqq 0$ and $\lambda \neq 0$, this shows that

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right)$$
$$+ w \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not holds for any $(A_1, ..., A_n) \in \mathscr{S}$. Q.E.D.

DEFINITION 3.3. A differentiable set function $F=(F_1, ..., F_m)\colon \mathscr{S} \to \mathbb{R}^m$ is said to be locally convex at $(\Omega_1, ..., \Omega_n)$ if there exists $\delta > 0$ such that

$$F(A_1, ..., A_n) \geqq F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right)$$

for all $(A_1, ..., A_n) \in \mathscr{S}$ with $d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) < \delta$.

The following theorem gives a sufficient conditions for the existence of a weak local minimum to problem (P).

THEOREM 5. *Suppose that the set function* $F=(F_1, ..., F_m)\colon \mathscr{S} \to \mathbb{R}^m$ *and* $G=(G_1, ..., G_p)\colon \mathscr{S} \to \mathbb{R}^p$ *are differentiable and locally convex at* $(\Omega_1, ..., \Omega_n)$. *If there exists* $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ *such that* $w(G(\Omega_1, ..., \Omega_n)) = 0$ *and*

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) + w\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not hold for any $(A_1, ..., A_n) \in \mathscr{S}$, *then* $(\Omega_1, ..., \Omega_n)$ *is a weak local minimum to* (P).

Proof. Let $w = (w_1, ..., w_m) \in B^+(\mathbb{R}^p, \mathbb{R}^m)$, then

$$w_i \in B^+(\mathbb{R}^p, \mathbb{R}^1) \qquad \text{for each} \quad i = 1, ..., m.$$

Let

$$H_j(A_1, ..., A_n) = \sum_{i=1}^{n} \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + w_j\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right).$$

It is easy to see that $H_j\colon \mathscr{S} \to \mathbb{R}^1$ is convex and the system

$$\begin{cases} H_1(A_1, ..., A_n) < 0 \\ \vdots \\ H_m(A_1, ..., A_n) < 0 \end{cases}$$

does not have a solution, then it follows from Theorem 2 that there exists nonzero

$$\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}_+^m$$

such that

$$\sum_{i=1}^{m} \lambda_i H_i(\Lambda_1, ..., \Lambda_n) \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}.$$

That is,

$$\begin{aligned}
&\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \\
&\qquad + \sum_{j=1}^{m} \lambda_j w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\
&\qquad \geqq 0, \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}. \qquad (10)
\end{aligned}$$

Since F and G are locally convex at $(\Omega_1, ..., \Omega_n)$, there exists $\delta > 0$ such that

$$\begin{aligned}
F(\Lambda_1, ..., \Lambda_n) &\geqq F(\Omega_1, ..., \Omega_n) \\
&\quad + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
G(\Lambda_1, ..., \Lambda_n) \geqq G(\Omega_1, ..., \Omega_n) + &\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\
&\text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S} \\
&\text{with} \quad d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta. \qquad (12)
\end{aligned}$$

By (10), (11), (12), and $w(G(\Omega_1, ..., \Omega_n)) = 0 = (w_1(G(\Omega_1, ..., \Omega_n)), ..., w_m(G(\Omega_1, ..., \Omega_n)))$, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \lambda_j [F_j(\Lambda_1, ..., \Lambda_n) - F_j(\Omega_1, ..., \Omega_n)] \\
&\qquad \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \\
&\qquad \geq - \sum_{j=1}^{m} \lambda_j w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\
&\qquad \geq - \sum_{j=1}^{m} \lambda_j w_j (G(\Lambda_1, ..., \Lambda_n)) + \sum_{j=1}^{m} \lambda_j w_j (G(\Omega_1, ..., \Omega_n)) \\
&\qquad = - \sum_{j=1}^{m} \lambda_j w_j (G(\Lambda_1, ..., \Lambda_n)) \\
&\qquad \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S} \\
&\qquad\qquad \text{with} \quad d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta \qquad (13)
\end{aligned}$$

Since $\lambda \geqq 0$, $\lambda \neq 0$, it follows from (13) that there exists no $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ with $G(\Lambda_1, ..., \Lambda_n) \leqq 0$ and $d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta$ such that

$$F(\Lambda_1, ..., \Lambda_n) < F(\Omega_1, ..., \Omega_n).$$

This shows that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to (P). Q.E.D.

The following corollary follows immediately from Theorems 3 and 5.

COROLLARY 6. *Let* $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ *be differentiable and locally convex at* $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$, *then* $(\Omega_1, ..., \Omega_n)$ *is a weak local minimum of* F *on* $\mathscr{S}$ *if and only if*

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not hold for any $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$.

Following a similar argument as in the proof of Theorem 4.5 [4], we have

LEMMA 7. *Let* $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ *be differentiable and convex on* $\mathscr{S}$, *then for all* $(\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n) \in \mathscr{S}$,

$$F(\Lambda_1, ..., \Lambda_n)) \geq F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right).$$

Remark. It follows from Lemma 7 that if $F: \mathscr{S} \to \mathbb{R}^m$ is differentiable and convex on $\mathscr{S}$, then F is locally convex at any $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$.

Applying Lemma 7 and following similar arguments as in the proof of Theorem 6, we have

THEOREM 8. *Suppose that the set function* $F: \mathscr{S} \to \mathbb{R}^m$ *and* $G: \mathscr{S} \to \mathbb{R}^p$ *are convex and differentiable on* $\mathscr{S}$. *If there exists* $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ *such that*

$$w(G(\Omega_1, ..., \Omega_n)) = 0$$

and

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + w\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not hold for any $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, *then* $(\Omega_1, ..., \Omega_n)$ *is a weak minimum to* (P).

REFERENCES

1. J. H. CHOU, W. S. HSIA, AND T. Y. LEE, On multiple objective programming problem with set functions, *J. Math. Anal. Appl.* **105** (1985), 383–394.
2. J. H. CHOU, W. S. HSIA, AND T. Y. LEE, Second order optimality conditions for mathematical programming with set functions, *J. Austral. Math. Soc. Ser. B* **26** (1985), 284–292.
3. J. H. CHOU, W. S. HSIA, AND T. Y. LEE, Epigraphs of convex set functions, *J. Math. Anal. Appl.* **118** (1986), 247–254.
4. H. W. CORLEY, Optimization for n-set functions, *J. Math. Anal. Appl.* **127** (1987), 193–205.
5. H. C. LAI AND L. J. LIN, The Fenchel–Moreau theorem for set functions, *Proc. Amer. Math. Soc.* **103** (1988), 85–90.
6. H. C. LAI AND L. J. LIN, Moreau–Rockafellar type theorem of convex set functions, *J. Math. Anal. Appl.* **132** (1988), 558–571.
7. H. C. LAI AND L. J. LIN, Optimality for set functions with values in ordered vector spaces, *J. Optim. Theory Appl.* **63** (1989), 405–423.
8. H. C. LAI, S. S. YANG, AND GEORGE R. HWANG, Duality in mathematical programming of set functions, on Fenchel duality theorem, *J. Math. Anal. Appl.* **95** (1983), 223–234.
9. H. C. LAI AND S. S. YANG, Saddle point and duality in the optimization theory of convex set functions, *J. Austral. Math. Soc. Ser. B* **24** (1982), 130–137.
10. D. G. LUENBERGER, "Optimization by Vector Space Methods," Wiley, New York, 1969.
11. R. J. T. MORRIS, "Optimization Problem Involving Set Functions," Ph.D. dissertation, University of California, Los Angeles, March 1978.
12. R. J. T. MORRIS, Optimal constrained selection of measurable subset, *J. Math. Anal. Appl.* **70** (1979), 546–562.
13. K. TANAKA AND Y. MARUYAMA, The multiobjective problem of set functions, *J. Inform. Optim. Sci.* **5** (1984), 293–306.